Analytic results for scaling function and moments for a different type of avalanche in the Bak-Sneppen evolution model

W. Li* and X. Cai†

Institute of Particle Physics, Hua-Zhong Normal University, Wuhan 430079, People's Republic of China

(Received 5 June 2000)

Starting from the master equation for the hierarchical structure of avalanches of a different kind within the frame of the Bak-Sneppen evolution model, we derive the exact formula of the scaling function describing the probability distribution of avalanches. The scaling function displays features required by the scaling ansatz and verified by simulations. Using the scaling function we investigate the avalanche moment, denoted by $\langle S^k \rangle_{\Delta \overline{f}}$. It is found that for any non-negative integer k, $\langle S^k \rangle_{\Delta \overline{f}}$ diverges as $\Delta \overline{f}^{-k}$, which gives an infinite group of exact critical exponents. Simulation outcomes of avalanche moments with $k=1,2,3$, are found to be consistent with the corresponding analytical results.

PACS number(s): 05.40.$-$a, 64.60.Ak, 87.10.$+$e

I. INTRODUCTION

The Bak-Sneppen (BS) evolution model [1] has become one of the most interesting models that display the nature of self-organized criticality (SOC) [2]. The BS model mimics the biological evolution in a very simple but most characterized way: L^d species are located on a d-dimensional lattice of linear size L. Initially, L^d random numbers chosen from a flat distribution between 0 and 1, $D(f)$, are assigned independently to each species as fitness. At each time step, the global extremal site, i.e., the species with the smallest fitness within the system, and its $2d$ nearest neighbors are assigned new random numbers also chosen from $D(f)$. After enormous updates the system reaches a statistically stationary state where the density of the fitnesses in the system is uniform above f_c (the self-organized threshold) and vanishes for $f < f_c$.

The BS model exhibits remarkably rich behaviors through the evolution based on an oversimplified rule, which is commonly shared by a wide class of extremal models [3]. These models evolve through the updating of global extremum of some variable and can be automatically attracted to the critical states by long processes of the so-called self-organization. In this class of models, the BS model may occupy a unique position since it is analytically treatable in many cases.

The most intriguing feature of the BS model is its ability to self-organize to a stationary critical state specified by a robust probability distribution of scale-free bursts of activity or avalanches. Larger avalanches consist of smaller ones and so on, which form a hierarchical structure, similar to those observed in fractals ubiquitous in nature [4]. It has been proposed by Paczuski, Maslov, and Bak [5] that the BS model and some other extremal models, together with many natural phenomena, can be unified on a common mechanism—avalanche dynamics, and that they are related via scaling relations to the fractal properties of the configurations carved by avalanches.

*Electronic address: liw@iopp.ccnu.edu.cn

†Electronic address: xcai@wuhan.cngb.com

As mentioned above, avalanche study plays a key role in comprehending the dynamics of the BS model. It is feasible to search for different hierarchy of avalanches in the BS model, whereas they manifest the same hierarchy—SOC—from various contexts [6]. Indeed, we have observed many different kinds of avalanches in the BS model [1,5,6]. In this paper, the avalanches originally defined in Ref. [1], f_0 avalanches [5] and $\overline{f_0}$ avalanches [6], will be called BS avalanches, PMB avalanches and LC avalanches, respectively.

As is known, in SOC models (e.g., the sandpile model [2], the BS model [1]), the probability distribution of avalanches of size S obeys a power law: $P(S)=S^{-\tau}\eta[F(S)]$, where the scaling function $\eta(x)$ decreases rapidly as $x \gg 1$ and approaches a constant as $x \to 0$. Up to now, despite that this scaling ansatz for various models has been verified by extensive simulations [1,7–11], the concrete form of $\eta(x)$ remains vague. This paper will show that for probability distribution of LC avalanches, the explicit form of the corresponding scaling function can be analytically derived.

In Sec. II, we briefly recall three different kinds of avalanches: BS avalanches, PMB avalanches, and LC avalanches. The scaling function for LC avalanches will be given in Sec. III. Avalanche moments for LC avalanches are investigated in Sec. IV. The last section is the conclusion.

II. THREE KINDS OF AVALANCHES

This section will review three different types of avalanches: BS avalanches, PMB avalanches, and LC avalanches. We will see that though these avalanches may differ from each other in some aspects, they all embody fundamental features of avalanches: compactness and hierarchical structure. In general, however, each kind of avalanche occupies its unique position.

A. Avalanches of the BS kind [1]

BS avalanches were observed when the BS model was first introduced [1]. Let the well-defined BS model start to evolve. With the evolution of the model, the lowest barrier, i.e., the smallest fitness within the system, tends to increase stepwisely. It is found that all mutations turn out to take place through barriers less than $f_c = 0.67 \pm 0.01$. Below some threshold smaller than or equal to f_c, long periods of passivity interrupted by sudden bursts of activity are observed. The punctuated equilibria emerge.

An effective way to characterize this intermittency is to monitor subsequent sequences, or avalanches, of mutations through barriers below a certain threshold. If there is no mutation for a time period defined by this threshold, the avalanches stop. The size S of an avalanche is the number of successive updates below the threshold. Thus, due to this definition, there is a hierarchy of avalanches, each defined by their respective thresholds. The critical exponent τ for the probability distribution of BS avalanches with threshold 0.65 is found to be 0.9 ± 0.1 for the 1D BS model.

B. Avalanches of the PMB kind [5]

The idea of f_0 avalanches, i.e., PMB avalanches, originated from BS avalanches. The only difference between them is that f_0 is introduced in the former. The revolutionary PMB avalanches greatly put forward our understanding of the dynamics of the evolution model.

f_0, between 0 and 1, is an auxiliary parameter in defining avalanches. In the critical state, PMB avalanches can be defined as follows. Suppose at time step s the fitness of the global extremal site in the ecosystem is greater than f_0. After an update, each new random numbers introduced at the same time step can be less than f_0 with probability f_0. If one or more random numbers is less than f_0, the smallest of those will be chosen for the next update at time step $s+1$. This update mechanism intrigues a creation-annihilation branching process where the species with fitness smaller than f_0 play the key role. If there is at least a species with fitness less than f_0 in the system at the consequential time step, the avalanche will continue. The avalanche terminates, say, at time step $S+s$, when the smallest fitness is above f_0 for the first time after time step s, where S is the size of the avalanche. The hierarchical structure of the PMB avalanches can be displayed by plotting the fitnesses of the global extremal sites versus the time steps.

The value of f_0 directly determines the probability distribution of PMB avalanches. When f_0 is not equal to f_c, the self-organized threshold, the statistics of PMB avalanches acquire a limited cutoff. The closer f_0 approaches to f_c, the larger the expected avalanche size will be. One can only expect to observe infinite avalanches when $f_0 = f_c$. And, it has been confirmed that a PMB avalanche in the stationary state is totally equivalent to the counterpart in the branching process [12]. This feature facilitates the simulations of PMB avalanches, which give the critical exponents τ for the probability distribution of PMB avalanches when $f_0 \rightarrow f_c$: 1.07 and 1.245 for 1D and 2D BS models, respectively.

C. Avalanches of the LC kind [7]

We have observed a different hierarchy of avalanches, $\overline{f}_0$ avalanches [6], called LC avalanches in this paper. Analytical investigations of LC avalanches have been presented in Refs. [13,14].

As is known, we intend to find a certain new quantity that can reflect the general features of the ecosystem, and expect that the avalanches can be observed through it. The average fitness $\overline{f}$ is what is being searched for. $\overline{f}$ is naturally introduced on the basis of the fitnesses of the species. It may represent the average population or living capability of the ecosystem. Larger $\overline{f}$ implies that the average population is immense or the average living capability is great, and vice versa. $\overline{f}$ is defined as

$$\overline{f} = \frac{1}{L^d} \sum_{i=1}^{L^d} f_i, \tag{1}$$

where f_i is the fitness of the ith species in the system. At each update of the evolution, apart from the random numbers of the globally extremal site and its $2d$ nearest-neighbor sites, the signal $\overline{f}$ is also monitored. Initially, $\overline{f}$ tends to increase like a staircase. As the evolution moves forward, $\overline{f}$ approaches a critical value $\overline{f}_c$ and remains statistically stable around $\overline{f}_c$ in the critical state. Numerical results give the values of $\overline{f}_c$: 0.834 and 0.664, for 1D and 2D models, respectively.

For any given value of the auxiliary parameter $\overline{f}_0 (0.5 < \overline{f}_0 < \overline{f}_c)$, an LC avalanche with $\overline{f}_0$ of size S is defined as a sequence of $S-1$ successive mutation events when $\overline{f}(s) < \overline{f}_0$ confined between two mutation events when $\overline{f}(s) > \overline{f}_0$. This definition properly embodies the spatiotemporal [15] features of LC avalanches. As $\overline{f}_0$ is raised, smaller avalanches merge into bigger ones, and as $\overline{f}_0$ decreases, larger avalanches split into smaller ones. The probability distribution of LC avalanches will have a limited cutoff when $\overline{f}_0$ is not chosen to be $\overline{f}_c$. However, when $\overline{f}_0$ is extremely close to $\overline{f}_c$, the statistics of the avalanches are good enough. Simulations show that critical exponents τ for the probability distribution of LC avalanches are 1.800 and 1.725 for 1D and 2D BS models, respectively [6], amazingly different than the counterparts of PMB avalanches, 1.07 and 1.245 [16].

III. SCALING FUNCTION FOR THE PROBABILITY DISTRIBUTION OF LC AVALANCHES

The hierarchical structure of LC avalanches can be prescribed by the ‘‘master equation’’ [17]. Denote $P(\overline{f}_0, S)$ as the probability of acquiring an LC avalanche with $\overline{f}_0$ of size S. As $\overline{f}_0$ is increased by an infinitesimal amount $d\overline{f}_0$, the signals $\overline{f}$'s which terminate the LC avalanches with $\overline{f}_0$ may not stop the LC avalanches with $\overline{f}_0 + d\overline{f}_0$. Hence, smaller avalanches will merge to form bigger ones. Straightforwardly, as $\overline{f}_0$ is decreased by some amount, larger avalanches split into smaller ones. This picture appropriately exhibits the hierarchical formations of the avalanches and should be embodied in the master equation. In some sense, the master equation reflects the flow of probability due to the change in $\overline{f}_0$, the parameter defining LC avalanches. Another important aspect is the distribution of signals $\overline{f}$'s that terminate LC avalanches. Simulations show that when $\overline{f}_0$ is very close to $\overline{f}_c$, the signals serving as breaking points of LC avalanches are uniformly distributed in $(\overline{f}_0,\overline{f}_c)$ [13].

Based on the above arguments, the probability of an LC avalanche with $\overline{f}_0$ of size S merging into an LC avalanche with $\overline{f}_0+d\overline{f}_0$ of size greater than S is given by $P(S,\overline{f}_0)d\overline{f}_0/(\overline{f}_c-\overline{f}_0)$. Therefore, as $\overline{f}_0$ is increased by an amount $d\overline{f}_0$, the ''emission'' (negative change) in $P(S,\overline{f}_0)$ is $P(S,\overline{f}_0)d\overline{f}_0/(\overline{f}_c-\overline{f}_0)$, whereas the ''absorption'' (positive change) in $P(S,\overline{f}_0)$ is $\Sigma_{S_0=1}^{S-1}P(S-S_0,\overline{f}_0)P(S_0,\overline{f}_0)d\overline{f}_0/(\overline{f}_c-\overline{f}_0)$. Considering the limit $d\overline{f}_0\to 0$, one can immediately write down the differential master equation for hierarchical structure of LC avalanches as follows [13]:

$$\partial_{\overline{f}_0}P(S,\overline{f}_0)=-(\overline{f}_c-\overline{f}_0)^{-1}P(S,\overline{f}_0)+\sum_{S_0=1}^{S-1}(\overline{f}_c-\overline{f}_0)^{-1}P(S-S_0,\overline{f}_0)P(S_0,\overline{f}_0), \tag{2}$$

where the first term on the right hand represents the loss of avalanches of size S due to the merging of subsequent ones, and the second one, the gain in $P(S,\overline{f}_0)$ due to the merging of avalanches of size S_0 with avalanches of size $S-S_0$.

The above master equation is of major interest. Based on the master equation, one can derive an infinite series of exact equations [13]. One can also write down two master equations for undercritical and overcritical LC avalanches hierarchy respectively, and make an analytical investigation for comprehending the features of undercritical and overcritical states [14]. These observations make us conjecture whether we can attain a further understanding of the above-all master equation and find some new intriguing features we have not anticipated. The answer is positive.

To make more inspections, let us return to the master equation. Note that the equation relates the probability distributions of LC avalanches for various $\overline{f}_0$. Apparently, imposing the initial condition $P(S,1/2)=\delta_{S,1}$ (i.e., 1 for $S=1$ and 0 for $S\geqslant 1$), one can readily obtain the probability distribution of LC avalanches by numerically integrating Eq. (2) forward in $\overline{f}_0$. In doing so, one acquires the exponent of the power-law distribution of LC avalanches and a by-product $\overline{f}_c$. We do not follow such a method, however, despite that it is feasible and may provide some useful information. Instead, we will provide an analytical method, which can also enable us to understand what we have attempted to without losing anything important.

Define

$$u=-\ln(\overline{f}_c-\overline{f}_0). \tag{3}$$

Since $\overline{f}_0$ is chosen to be very close to $\overline{f}_c$, the variable u varies from a finite large number u_0 to $u_c=+\infty$. Due to this, the value of u should be taken from the distribution of e^{-u}, a more ''natural'' distribution, in contrast to the ''uniform'' one. The master equation for LC avalanches hierarchy can be rewritten, in terms of u, as,

$$\partial_u P(S,u)=-P(S,u)+\sum_{S_0=1}^{S-1}P(S-S_0,u)P(S_0,u). \tag{4}$$

Laplace transformation of $P(S,u)$, i.e., $p(\alpha,u)=\Sigma_{S=1}^{\infty}P(S,u)e^{-\alpha u}$, on both sides of Eq. (4), leads to the following equation:

$$\partial_u p(\alpha,u)=p^2(\alpha,u)-p(\alpha,u). \tag{5}$$

As seen from Eq. (5), the original coupled master equation [Eq. (2)] now takes on a simple form.

It has been mentioned that if $\overline{f}_0$ is not equal to $\overline{f}_c$, the probability distribution of LC avalanches will have a limited cutoff. In analogy to ordinary percolation [18] and directed percolation [19], we adopt the following scaling ansatz for the probability distribution $P(S,\overline{f}_0)$ of LC avalanches of size S,

$$P(S,\overline{f}_0)=S^{-\tau}H(S^{\sigma}(\overline{f}_c-\overline{f}_0)), \tag{6}$$

where the model dependent exponent σ describes the cutoff of LC avalanches and is found to be 0.200 and 0.275 for 1D and 2D BS models, respectively [13]. Here, the scaling function $H(x)$ approaches a constant when $x\to 0$ and decays rapidly when $x\gg 1$. This scaling ansatz has been confirmed by our simulations.

With the scaling ansatz [Eq. (6)] one can write down the expression of $p(\alpha,u)$ as

$$p(\alpha,u)=\sum_{S=1}^{\infty}e^{-\alpha S}S^{-\tau}H(S^{\sigma}e^{-u}). \tag{7}$$

Here, $\overline{f}_0$ has been replaced by u. Replacing the sum with the integral in Eq. (7) (when $S\to+\infty$), after some algebra, one obtains

$$p(\alpha,u)=1-\alpha^{\tau-1}g(e^{-u}\alpha^{-\sigma}), \tag{8}$$

where the scaling function $g(x)$ is related to $H(x)$ by

$$g(x)=\int_0^{+\infty}[H(0)-H(xy^{\sigma})]e^{-y}y^{-\tau}dy. \tag{9}$$

Hence, if we understand features of the scaling function $g(x)$, we will understand those of $H(x)$.

Define

$$z=e^{-u}\alpha^{-\sigma}. \tag{10}$$

Hence, $u_c=+\infty$ corresponds to $z_c=0$. Combination of Eq. (5) and Eq. (8) results in a differential equation of $g(z)$,

$$zg'(z)=\alpha^{\tau-1}g^2(z)-g(z). \tag{11}$$

The solution of Eq. (11) gives the explicit form of the scaling function $g(z)$,

$$g(z)=(z+\alpha^{\tau-1})^{-1}. \tag{12}$$

As one can see, the expression of $g(z)$ has the desired asymptotic form. Define

$$g(z)=(z-z_c)^{-\beta_s}, \tag{13}$$

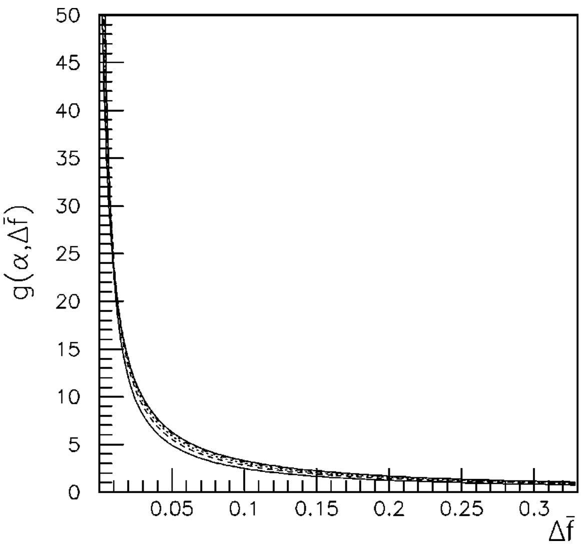

FIG. 1. The scaling function $g(\alpha,\Delta\overline{f})$ vs $\Delta\overline{f}$ for various α. α is taken to be 0.001, 0.002, 0.003, 0.004, and 0.005, respectively.

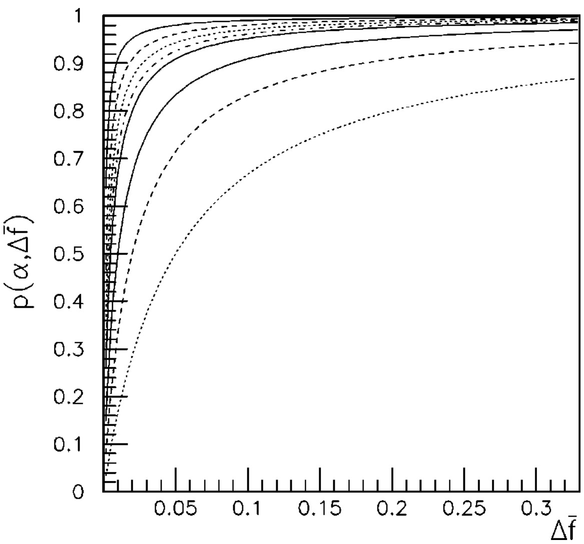

FIG. 2. $p(\alpha,\Delta\overline{f})$ vs $\Delta\overline{f}$. Here, the values of α for the curves from the bottom to the top are 0.001, 0.002, 0.003, 0.004, 0.005, 0.01, 0.02, and 0.03, respectively.

where we call β_s the scaling exponent. As $\alpha\to 0$, $g(z)$ behaves like z^{-1}, i.e., $(z-z_c)^{-1}$, which leads to

$$\beta_s=1. \tag{14}$$

The scaling exponent β_s reflects the tendency of divergence of $g(z)$ near the critical state.

By changing variable z back into $\Delta\overline{f}=\overline{f_c}-\overline{f_0}$ one obtains

$$g(\alpha,\Delta\overline{f})=(\Delta\overline{f}\alpha^{-\sigma}+\alpha^{-1})^{-1}. \tag{15}$$

One sees that $g(\alpha,\Delta\overline{f})$ goes to a constant $1/\alpha^{\tau-1}$ as the critical state is approached, and decreases rapidly as $\overline{f_0}$ does. Since in the BS model (1D and 2D as well) the exponent τ for LC avalanches size distribution is greater than 1 [6], one can expect that for larger α, the scaling function $g(\alpha,\Delta\overline{f})$ decreases more slowly. These features have been clearly demonstrated in Fig. 1, where we plot $g(\alpha,\Delta\overline{f})$ versus $\Delta\overline{f}$ for various α. What this figure manifests is consistent with what we had expected for the scaling function.

Making use of the formula of the scaling $g(z)$, one can immediately obtain the expression of $p(\alpha,u)$,

$$p(\alpha,u)=1-\frac{\alpha^{\tau-1}}{e^{-u}\alpha^{-\sigma}+\alpha^{\tau-1}}. \tag{16}$$

Considering the relation between u and $\overline{f_0}$, one sees that $p(\alpha,u)$ decreases as $\overline{f_0}$ increases, which means that $p(\alpha,u)$ attains its maximum at the first beginning of the evolution. For clarity, we plot $p(\alpha,\Delta\overline{f})$ versus $\Delta\overline{f}$ for various α in Fig. 2. As shown, when α is taken to be 0, the normalization of the probability distribution of LC avalanches, i.e., $p(0,u)=\Sigma_0^\infty P(S,u)=1$, is automatically restored.

IV. AVALANCHE MOMENTS FOR LC AVALANCHES

Evidently, the scaling function $g(x)$ [as well as $H(x)$] is analytical at $x=0$. At any $u<u_c=+\infty$, $p(\alpha,u)$ is analytical in α. Expansion of the expression of $p(\alpha,u)$ through the neighborhood of $\alpha=0$ yields

$$\sum_{S=0}^{\infty}\frac{(-1)^k}{k!}\langle S^k\rangle_u\bigg|_{\alpha=0}\alpha^k=\sum_{S=0}^{\infty}\frac{1}{k!}p^{(k)}(\alpha,u)\bigg|_{\alpha=0}\alpha^k, \tag{17}$$

where $p^{(k)}(\alpha,u)$ denotes the kth-order derivative of $p(\alpha,u)$ over α and $\langle S^k\rangle_u$ is the kth-order avalanche moment for LC avalanches. Since Eq. (17) holds for arbitrary α, comparison of coefficients of different powers of α results in an infinite hierarchy of exact equations. Comparing coefficient of α^k gives the kth-order avalanche moment

$$\langle S^k\rangle_u=(-1)^k p^{(k)}(\alpha,u)|_{\alpha=0}. \tag{18}$$

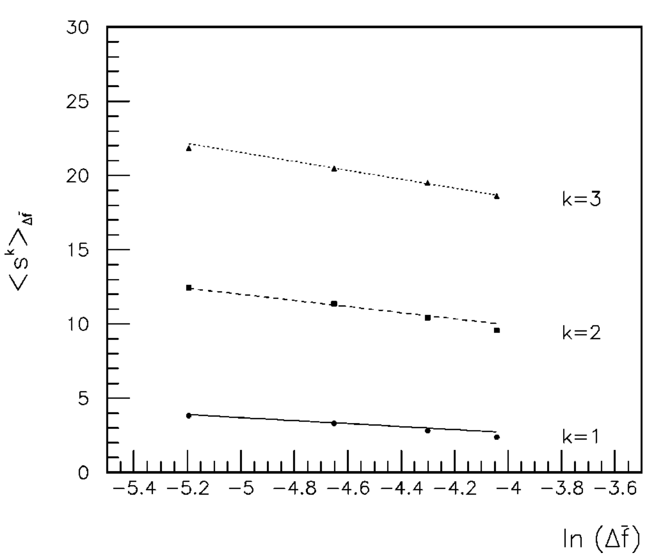

FIG. 3. $\langle S^k\rangle_{\Delta\overline{f}}(k=1,2,3)$ vs $\Delta\overline{f}$ for LC avalanches in 1D BS model. The respective slopes are -1.01, -2.01, and -3.02.

For $k=0$, the normalization of the probability distribution of LC avalanches is restored. For $k=1$, one obtains

$$\langle S\rangle_{\Delta\overline{f}}\sim(\overline{f_c}-\overline{f_0})^{-1}, \tag{19}$$

which has been confirmed theoretically and numerically [13]. Imposing the scaling relation $\sigma+\tau=2$ [14] into Eq. (18) and changing u back into $\Delta\overline{f}$, one readily obtains an amazing result,

$$\langle S^k\rangle_{\Delta\overline{f}}\sim(\overline{f_c}-\overline{f_0})^{-k}. \tag{20}$$

This equation shows that avalanche moments diverge more quickly with higher integer k. This feature can be directly seen from the following form:

$$\langle S^k\rangle_{\Delta\overline{f}}\sim\int_0^{(\overline{f_c}-\overline{f_0})^{-1/\sigma}}S^{-\tau}S^k dS. \tag{21}$$

Introducing

$$\langle S^k\rangle\sim(\overline{f_c}-\overline{f_0})^{-\gamma_k}, \tag{22}$$

where γ_k is the critical exponent governing the divergence of the kth-order avalanche moment, and comparing it with Eq. (20) yields

$$\gamma_k=k. \tag{23}$$

γ_k's for various k form an infinite group of exponents that specify the behaviors of avalanche moments of various orders near the critical point. To check such results, simulations of avalanche moments with $k=1,2,3$ for LC avalanches are performed. The critical exponents obtained from the simulations are in good agreement with the corresponding analytical results, as shown in Fig. 3.

V. CONCLUSION

Compared to PMB avalanches, the newly observed LC avalanches are more easily treated analytically. This is true since many important exponents, e.g., γ, ρ [13], and σ, which specify the dynamical features of the avalanches hierarchy, and the scaling function, for LC avalanches can be exactly obtained through the fundamental master equation. Although one can also treat PMB avalanches analytically [17], it appears more difficult. This argument, however, does not intend to deny the importance of PMB avalanches in the BS model. On the contrary, each kind of avalanches remains unique. Only by combining all these different avalanches could it be possible for one to comprehend the features of the model more completely. It appears feasible for one to investigate the relationship between PMB avalanches and LC avalanches, since both of them obey the power law. What intrinsic relation lies between them is intriguing and interesting.

It can be readily inferred that the master equation does play an important role in the BS model. It can be regarded as the fundamental rule since it is written down just according to the conservation of the probability distribution of avalanches, without any additional ansatz, while many interesting features of the avalanche hierarchy can be derived from the equation. This make us conjecture whether we can find such counterparts in other extremal models. If it is the case, we may gain some new observations.

In summary, we derive the explicit form of the scaling function for the probability distribution of LC avalanches by using the master equation. The asymptotic behavior of the scaling function is in consistency with the scaling ansatz and simulations. In the use of the scaling function, we investigate the behaviors of avalanche moments of various orders. It is found that the critical exponents governing the divergence of avalanche moments of various orders form an infinite group $\{1,2,3,\ldots,k,\ldots\}$. Simulations of avalanche moments with $k=1,2,3$ verify the corresponding analytical results.

ACKNOWLEDGMENTS

This work was supported in part by the NSFC and Hubei-NSF in China.

[1] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[2] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[3] S.I. Zaitsev, Physica A **189**, 411 (1992); K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992); L.-H. Tang and H. Leschhorn, *ibid.* **70**, 3832 (1993).
[4] B.B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).
[5] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[6] W. Li and X. Cai, Phys. Rev. E **61**, 771 (2000).
[7] T.S. Ray and N. Jan, Phys. Rev. Lett. **72**, 4045 (1994).
[8] J. de Boer, B. Derrida, H. Flyvbjerg, A.D. Jackson, and T. Wettig, Phys. Rev. Lett. **73**, 906 (1994); J. de Boer, A.D. Jackson, and T. Wettig, Phys. Rev. E **51**, 1059 (1995).
[9] L.-H. Tang and H. Leschhorn, Phys. Rev. Lett. **70**, 3832 (1993); H. Leschhorn and L.-H. Tang, Phys. Rev. E **51**, 1059 (1995).
[10] N. Martys, M.O. Robbins, and M. Cieplak, Phys. Rev. B **44**, 12 294 (1991).
[11] R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **52**, 1547 (1984); H. Ji and M.O. Robbins, Phys. Rev. B **46**, 14 519 (1992).
[12] M. Paczuski, S. Maslov, and P. Bak, Europhys. Lett. **27**, 97 (1994).
[13] W. Li and X. Cai, Phys. Rev. E **61**, 5630 (2000).
[14] W. Li and X. Cai (unpublished).
[15] Y.-C. Zhang, J. Phys. I **1**, 971 (1991); H.C. Fogedby, J. Stat. Phys. **69**, 411 (1992).
[16] P. Grassberger, Phys. Lett. A **200**, 277 (1995); B. Jovanovic, S.V. Buldyrev, A. Havlin, and H.E. Stanley, Phys. Rev. E **50**, 2403 (1993).
[17] S. Maslov, Phys. Rev. Lett. **77**, 1182 (1996).
[18] D. Stauffer, *Introduction to Percolation Theory* (Taylor, London, 1985); G. Grimmett, *Percolation* (Springer, New York, 1989).
[19] J. Feder, *Fractals* (Plenum, New York, 1989).